Development and verification of a mathematical model of a simple electromagnetic train

What is the relationship between various physical constants of an electromagnetic “train” and “track” and the terminal velocity of the train?

Physics

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Contents

Essay Overview

Electromagnetism is arguably the most engaging and entertaining field of physics for the layman, and it thus has great potential to inspire the general population to learn more about this fundamental interaction of the universe. One application of the study of electromagnetism which has captivated millions is the "world's simplest electric train" as showcased in a viral video online (AmazingScience, 2014). Although this phenomenon has been modeled previously (Criado & Alamo, 2016), there is still a need to deepen our understanding of this amusing demonstration of electromagnetic induction. To build on the work of the physics community in providing a mathematical model of this interaction, this paper will rederive the model developed by Criado and Alamo in a way more accessible to those without prior knowledge of electromagnetism and verify the accuracy of the model by solving for the train's terminal velocity and comparing it to empirical data.

This essay aims to answer the question, **"What is the relationship between various physical constants of an electromagnetic 'train' and 'track' and the terminal velocity of the train?"**. The "train" refers to the arrangement of spherical magnets, a AA battery, and a washer attached to a hanging weight which moves through a left-handed coil of copper wire called the "track", as shown in *Picture 1*.

Picture 1: Track, train, and attached weight

The physical constants mentioned in the question refer to the internal radius of the coil, the average number of turns per meter of coil, the distance between the centers of the magnets, the radii of the magnets, the magnitude of the magnetic moments of the magnets, the mass of the train, the voltage of the battery, the coefficient of kinetic friction of the system, the total resistance of the circuit, and the mass of the hanging weight.

In the explanation of the phenomenon, this paper will apply the concepts of electromagnetic induction, eddy currents, and Newton’s laws of motion.

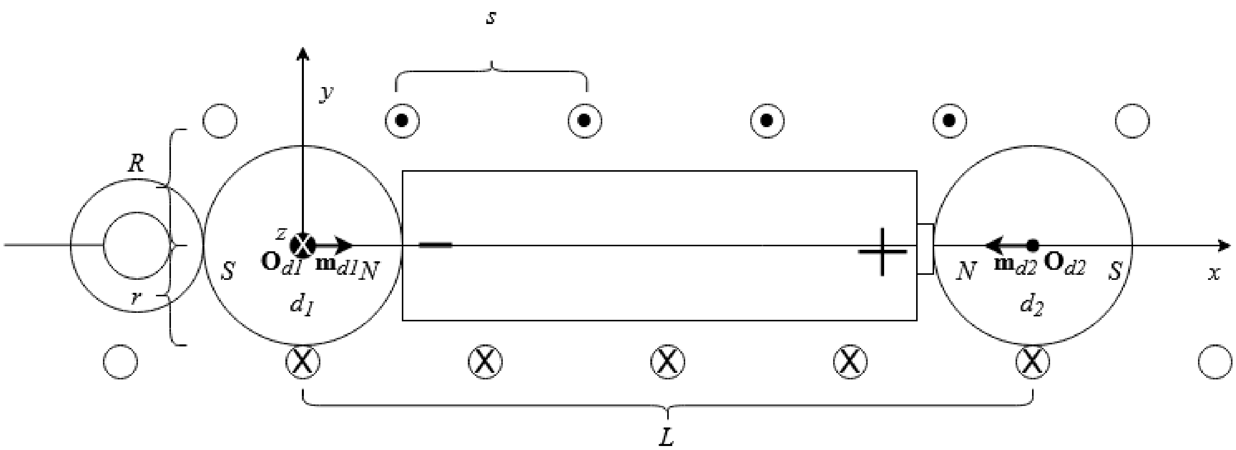
When the two magnets of the train contact the coil, current flows through the circuit formed by the battery and the coil, generating a magnetic field. This magnetic field interacts with the magnets, producing a force acting on the train. As the train moves through the coil, the changing flux of either magnet’s magnetic field through the coil generates a current opposing the current provided by the battery, decreasing the force acting on the train. Since the change in flux the coil is exposed to over a change in time is dependent on the velocity of the train which is in turn dependent on the strength of the magnetic field generated by the current in the coil, the velocity of the train is self-limiting, and the train will eventually reach a terminal velocity which can be calculated given the above physical characteristics of the system. The purpose of the washer and attached weight is to make data collection less cumbersome, as explained in the section regarding the experimental procedure, and the effect of the washer on the magnetic field generated by the trailing magnet will be assumed to be negligible.

The first step in the construction of the model will be the calculation of the electromotive force caused by the movement of the two magnets through the coil which will later be used to find the eddy current generated by the change in magnetic flux. The value of the eddy current will then be used in the calculation of the magnetic field produced by the flow of current through the coil and how this magnetic field interacts with the magnets of the train. Once the force of friction and the force of the hanging weight have been factored into the calculations, we will be able to isolate the velocity of the train to produce the desired equation.

Definition of Symbols

Throughout this investigation, all vector quantities will be represented as bolded, unitalicized variables. Their magnitudes will be represented as the same symbol with italicization but without bolding (e.g. vector **u** would have magnitude u), and their components will be represented by the same symbol as their magnitude with a subscript denoting which component it represents (e.g. vector **u** would have components $\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$). The symbols to be used throughout this investigation are depicted in *Diagram 1*. The coordinate system used to describe the motion of the train will consist of an *x*-axis running longwise through the coil; a *y*-axis opposite the pull of gravity; and a *z*-axis perpendicular to the plane formed by the *x*- and *y*-axes and pointing leftwards from the positive x-direction, "into" the page. The crossed and dotted circles indicate where the current is flowing "into" and "out of" the page, respectively. The distance $L = 0.0890$ m between the centers of the dipoles and the average separation between turns $s = \frac{L}{N} = 0.00254$ m where $N = 35.0$ is the average number of turns over distance L are also presented in the diagram. Magnetic dipoles d_1 and d_2 are located at $\mathbf{O}_{d_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{O}_{d_2} = \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix}$ respectively; it is assumed that the train sits perfectly in the center of the coil. The magnetic moment is a vector quantity with a magnitude equal to the torque the dipole experiences in unit magnetic field and in the direction pointing from the south pole of the dipole to its north pole (Tatum, 2020). The magnetic moment of dipole d_1 is $\mathbf{m}_{d_1} = \begin{bmatrix} m \\ 0 \\ 0 \end{bmatrix}$, whereas that of dipole d_2 is $\mathbf{m}_{d_2} = \begin{bmatrix} -m \\ 0 \\ 0 \end{bmatrix}$, where m is the common magnitude of each dipole's magnetic moment.

Diagram 1: Train and track configuration



For a spherical magnet of radius r and uniform magnetization $\mathbf{M}$, its internal magnetic field is given by $B = \frac{2}{3}\mu_0 M$ (Griffiths, 2018, p. 276). Rearranging for M,

$$M = \frac{3B}{2\mu_0}. \tag{2.1}$$

The external magnetic field of a spherical magnet is a pure dipole with magnetic moment

$$m = \frac{4}{3}\pi r^3 M \tag{2.2}$$

where r is the radius of the dipole (Griffiths, 2018, p. 276). Substituting equation 2.1 into equation 2.2, $m = \frac{2\pi r^3}{\mu_0}B$. For the magnets I used in the experiment, the magnitude of their internal magnetic fields is $B = 1.32$ T (K&J Magnetics, Inc., 2020). Using a value of $r = 0.009525$ (K&J Magnetics, Inc., 2020), each magnet has a magnetic moment with magnitude $m = 5.70$ A m^2.

Calculation of the Electromotive Force

Magnetic flux is the term used to describe the amount of magnetic field going through a given area (Urone & Hinrichs, 2020). When the magnetic flux present in a conductor changes, small loops of current called eddy currents are generated. These eddy currents flow in loops which generate magnetic fields with magnetic moments opposite those of the magnetic field which caused the change in magnetic flux. In the case of the train and track, the magnetic fields of the two dipoles move relative to the coil, creating eddy currents in the coil. Since the current and consequently the force generated by this interaction is proportional to the velocity of the magnetic field relative to the coil, the velocity of the magnetic field is self-limiting; this implies that the train will reach a terminal velocity at which the force on the train due to the magnetic field generated by the helical current resulting from the contributions of the battery and the eddy currents will be equal to the combined force of friction and the weight opposing its motion (Ling, Sanny, & Moebs, 2020).

Since the eddy currents can only be generated in a closed circuit, the only section of coil which will be considered in the calculations is that which lies between either dipole. Given the voltage V of the circuit, the current through the circuit I, and the internal resistance of the battery R_I, the current of the circuit is given by $I = \frac{\varepsilon - V}{R_I}$, where ε is the electromotive force (emf) acting on the system (Fitzpatrick, Emf and Internal Resistance, 2007). By Ohm's law, $V = IR_C$ where R_C is the resistance of the circuit excluding the battery. Substituting this into the above expression and rearranging for I,

$$I = \frac{\varepsilon}{R_T}, \tag{3.1}$$

where $R_T = R_C + R_I$ is the total resistance of the circuit. This equation can be used to find the eddy current I_E relevant to the problem. Applying Lenz's Law,

$$\varepsilon = -\frac{d\phi}{dt} \tag{3.2}$$

(Griffiths, 2018, p. 318) where ϕ is the magnetic flux of the combined magnetic field of dipole d_1 and d_2 through the helicoid H. The helicoid is defined by

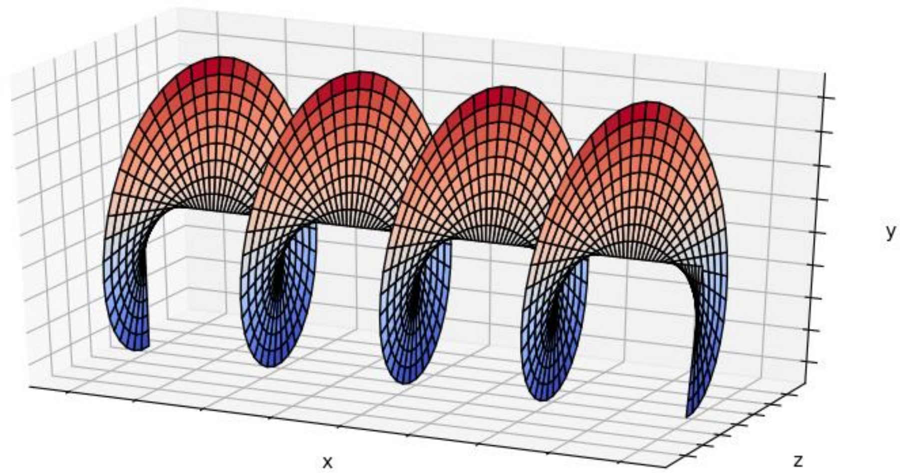
$$\mathbf{q} = \begin{bmatrix} x \\ -\rho\cos(kx) \\ -\rho\sin(kx) \end{bmatrix}$$

for the horizontal position $0 \leq x \leq L$ and the radius of the helicoid $0 \leq \rho \leq R$, where

$$k = \frac{2\pi}{s} = \frac{2\pi N}{L} \tag{3.3}$$

translates into one turn being completed for every change in x equivalent to the average separation between turns, with $s = \frac{L}{N}$ being the average separation between turns where N is the average turns within the length L. When $x = 0$, $\mathbf{q} = \begin{bmatrix} 0 \\ -\rho \\ 0 \end{bmatrix}$, indicating the bottom of the helix. As x increases, the y- and z-components trace a spiral clockwise when viewed from the positive x-direction, forming the desired shape, as depicted in *Figure 1*.

Figure 1: Helicoid over which the flux is calculated



The magnetic flux of the combined field $\mathbf{B}$ produced by dipoles d_1 and d_2 through the helicoid is given by

$$\phi = \int_H \mathbf{B} \cdot d\mathbf{A} \tag{3.5}$$

where $d\mathbf{A}$ is the infinitesimal area of the helicoid over which the magnetic flux is calculated, as given by

$$d\mathbf{A} = \left(\frac{\partial \mathbf{q}}{\partial x} \times \frac{\partial \mathbf{q}}{\partial \rho}\right) dx d\rho$$

(Purcell & Morin, 2013, p. 350) where $\frac{\partial \mathbf{q}}{\partial x} = \frac{\partial}{\partial x}\begin{bmatrix} x \\ -\rho\cos(kx) \\ -\rho\sin(kx) \end{bmatrix} = \begin{bmatrix} 1 \\ k\rho\sin(kx) \\ -k\rho\cos(kx) \end{bmatrix}$ and $\frac{\partial \mathbf{q}}{\partial \rho} = \frac{\partial}{\partial \rho}\begin{bmatrix} x \\ -\rho\cos(kx) \\ -\rho\sin(kx) \end{bmatrix} = \begin{bmatrix} 0 \\ -\cos(kx) \\ -\sin(kx) \end{bmatrix}$, giving

$$d\mathbf{A} = \left(\begin{bmatrix} 1 \\ k\rho\sin(kx) \\ -k\rho\cos(kx) \end{bmatrix} \times \begin{bmatrix} 0 \\ -\cos(kx) \\ -\sin(kx) \end{bmatrix}\right) dx d\rho = \begin{bmatrix} -k\rho\sin^2(kx) - k\rho\cos^2(kx) \\ \sin(kx) \\ -\cos(kx) \end{bmatrix}$$

$$= \begin{bmatrix} -k\rho(\sin^2(kx) + \cos^2(kx)) \\ \sin(kx) \\ -\cos(kx) \end{bmatrix}.$$

Recognizing the Pythagorean identity in dA_x, this expression can be further reduced to

$$d\mathbf{A} = \begin{bmatrix} -k\rho \\ \sin(kx) \\ -\cos(kx) \end{bmatrix} d\rho dx. \tag{3.6}$$

The magnetic field of a dipole of moment $\mathbf{m}$ at a point $\mathbf{b}$ relative to the position of the dipole is given by

$$\mathbf{B}(\mathbf{m}, \mathbf{b}) = \frac{\mu_0}{4\pi b^3}\left[3(\mathbf{m} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}} - \mathbf{m}\right] \tag{3.7}$$

where

$$\hat{\mathbf{b}} = \frac{\mathbf{b}}{b} \tag{3.8}$$

(Griffiths, 2018, p. 255). Since the x-component B_x of the combined field $\mathbf{B}$ is the only dimension aligned with the coil, it will be the only contributor to the eddy currents, and we can ignore the other dimensions of $\mathbf{B}$. B_x is given by

$$B_x(\mathbf{m}, \mathbf{b}) = \frac{\mu_0}{4\pi b^3}\left[3(\mathbf{m} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}} - \mathbf{m}\right]_x. \tag{3.9}$$

In the context of the problem, $\mathbf{b}$ will follow the form

$$\mathbf{q} = \begin{bmatrix} x \\ -\rho\cos(kx) \\ -\rho\sin(kx) \end{bmatrix}$$

in accordance with equation 3.4. As a result, B_x can be rewritten as

$$B_x(\mathbf{m}, x, \rho) = \frac{\mu_0}{4\pi q^3}[3(\mathbf{m}\cdot\hat{\mathbf{q}})\hat{\mathbf{q}} - \mathbf{m}]_x \tag{3.10}$$

Because dipole d_2 is offset from dipole d_1 by L across the x-axis, a point located at $\mathbf{q}$ relative to dipole d_1 will be located at $\mathbf{q} - \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix}$ relative to dipole d_2. Using equation 3.10, the combined x-component of the field **B** of the fields of dipoles d_1 and d_2 can be found with

$$B_x(x, \rho) = B_x(\mathbf{m}_{d_1}, x, \rho) + B_x(\mathbf{m}_{d_2}, x - L, \rho). \tag{3.11}$$

Substituting this equation into equation 3.5,

$$\phi = \int_H B_x(x, \rho) \cdot dA_x.$$

Expanding the integral to cover the helicoid defined by equation 3.3 and substituting in the value of dA_x given by equation 3.6,

$$\phi = \int_0^L \int_0^R B_x(x, \rho) \cdot -k\rho \, d\rho \, dx$$

$$= -k \int_0^L \int_0^R B_x(x, \rho)\rho \, d\rho \, dx.$$

Substituting this equation into equation 3.2,

$$\varepsilon = \frac{d}{dt}\left[k\int_0^L\int_0^R B_x(x,\rho)\rho\,d\rho\,dx\right]$$

$$= k\int_0^L\int_0^R \frac{d}{dt}\big(B_x(x,\rho)\big)\rho\,d\rho\,dx. \tag{3.12}$$

As the train moves in the positive x-direction relative to the coil, the coil moves in the negative x-direction relative to the train at the same speed. Thus, the horizontal position of a point on the helicoid relative to dipole d_1 is given by

$$x(t) = x - vt \tag{3.13}$$

where x is the initial x-position of the point relative to dipole d_1. When the train is positioned such that dipole d_1 is centered at $x = 0$ at time $t = 0$ and the train moves at a constant velocity v in the x-direction, we can substitute equation 3.13 in place of x in equation 3.12, yielding

$$\varepsilon = k\int_0^L\int_0^R \frac{d}{dt}\big(B_x(x(t),\rho)\big)\rho\,d\rho\,dx.$$

Using the chain rule, this becomes

$$\varepsilon = k\int_0^L\int_0^R \frac{dB_x(x,\rho)}{dx}\cdot\frac{dx(t)}{dt}\rho\,d\rho\,dx. \tag{3.14}$$

From the definition of $x(t) = x - vt$ given by equation 3.13, it is clear that

$$\frac{dx(t)}{dt} = -v.$$

Substituting this into equation 3.12,

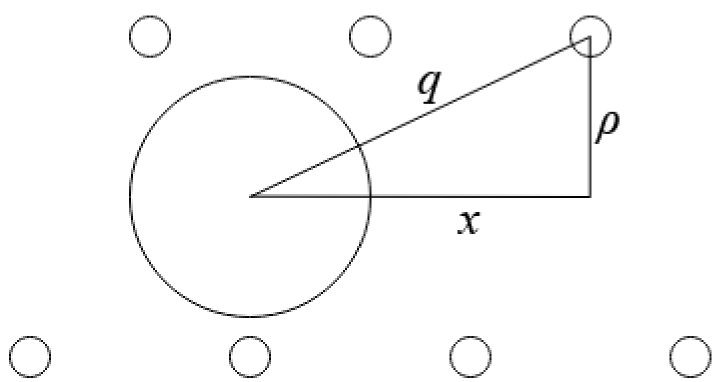
$$\varepsilon = -vk \int_0^L \int_0^R \frac{dB_x(x,\rho)}{dx} \rho \, d\rho \, dx. \tag{3.15}$$

Expanding $B_x(x)$ with its definition in equation 3.11,

$$B_x(x,\rho) = B_x(\mathbf{m}_{d_1}, x, \rho) + B_x(\mathbf{m}_{d_2}, x - L, \rho). \tag{3.16}$$

To begin expanding these expressions, we must derive $\hat{\mathbf{q}}$ given equations 3.8 and 3.4. Since each dipole is aligned with the center of the helicoid in the y- and z-directions, the vector $\mathbf{q}$ from the center of a dipole to a point on the helicoid defined by equation 3.4 has a magnitude $q = \sqrt{x^2 + \rho^2}$ when $q_x = x$, as clarified by _Diagram 2_.

Diagram 2: Clarification of geometry



Subsequently, when $\mathbf{q}$ is given by equation 3.8,

$$\hat{\mathbf{q}} = \frac{\mathbf{q}}{q} = \begin{bmatrix} \frac{x}{q} \\ \frac{-\rho\cos(kx)}{q} \\ \frac{-\rho\sin(kx)}{q} \end{bmatrix}, \tag{3.17}$$

where

$$q = \sqrt{x^2 + \rho^2}. \tag{3.18}$$

Since $\mathbf{m}_{d_1} = \begin{bmatrix} m \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{m}_{d_2} = \begin{bmatrix} -m \\ 0 \\ 0 \end{bmatrix}$, equation 3.10 can be simplified in the context of this problem to only require the magnitude of the magnetic moment of the dipole rather than the moment expressed as a vector along with the location of the point in the x-direction. Given a magnitude γ and using the definition of $\hat{\mathbf{q}}$ provided by equation 3.17, equation 3.10 becomes

$$B_x\,(\gamma, x, \rho) = \frac{\mu_0}{4\pi q^3}\left[3\left(\begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{x}{q} \\ \frac{-\rho\cos(kx)}{q} \\ \frac{-\rho\sin(kx)}{q} \end{bmatrix}\right)\begin{bmatrix} \frac{x}{q} \\ \frac{-\rho\cos(kx)}{q} \\ \frac{-\rho\sin(kx)}{q} \end{bmatrix} - \begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix}\right]_x$$

$$= \frac{\mu_0}{4\pi q^3}\left[\frac{3\gamma x}{q}\begin{bmatrix} \frac{x}{q} \\ \frac{-\rho\cos(kx)}{q} \\ \frac{-\rho\sin(kx)}{q} \end{bmatrix} - \begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix}\right]_x$$

$$= \frac{\mu_0}{4\pi q^3}\left[\begin{bmatrix} \frac{3\gamma x^2}{q^2} \\ \frac{-3\gamma x\rho\cos(kx)}{q^2} \\ \frac{-3\gamma x\rho\sin(kx)}{q^2} \end{bmatrix} - \begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix}\right]_x$$

$$= \frac{\mu_0}{4\pi} \frac{\begin{bmatrix} \frac{3\gamma x^2 - \gamma q^2}{q^2} \\ \frac{-3\gamma x\rho\cos(kx)}{q^2} \\ \frac{-3\gamma x\rho\sin(kx)}{q^2} \end{bmatrix}_x}{q^3}$$

$$= \frac{\mu_0}{4\pi} \frac{3\gamma x^2 - \gamma q^2}{q^5}$$

$$= \frac{\mu_0 \gamma}{4\pi} \frac{3x^2 - q^2}{q^5}.$$

Substituting in equation 3.18 for q, this becomes

$$B_x\,(\gamma, x, \rho) = \frac{\mu_0 \gamma}{4\pi} \frac{3x^2 - (x^2 + \rho^2)}{(x^2 + \rho^2)^{\frac{5}{2}}}$$

$$= \frac{\mu_0 \gamma}{4\pi} \frac{2x^2 - \rho^2}{(x^2 + \rho^2)^{\frac{5}{2}}}. \tag{3.19}$$

Using this generalized formula, the total field in the x-direction is given by

$$B_x(x, \rho) = B_x(m, x, \rho) + B_x(-m, x - L, \rho). \tag{3.20}$$

By the sum rule,

$$\frac{dB_x(x, \rho)}{dx} = \frac{dB_x(m, x, \rho)}{dx} + \frac{dB_x(-m, x - L, \rho)}{dx} \tag{3.21}$$

To calculate ε, we must first calculate $\frac{dB_x(x,\rho)}{dx}$. This can be done by calculating $\frac{dB_x(\gamma,x,\rho)}{dx}$ and applying the result to both terms of equation 3.21. Finding the derivative of equation 3.19,

$$\frac{dB_x(\gamma,x,\rho)}{dx} = \frac{d}{dx}\left[\frac{\mu_0\gamma}{4\pi}\frac{2x^2-\rho^2}{(x^2+\rho^2)^{\frac{5}{2}}}\right]$$

$$= \frac{\mu_0\gamma}{4\pi}\cdot\frac{d}{dx}\left[\frac{2x^2-\rho^2}{(x^2+\rho^2)^{\frac{5}{2}}}\right] \tag{3.22}$$

The chain rule states that, given a function $f(x) = g(h(x))$,

$$f'(x) = g'\big(h(x)\big)\cdot h'(x).$$

In equation 3.22, $f(x) = (x^2+\rho^2)^{\frac{5}{2}}$, $g(x) = x^{\frac{5}{2}}$, and $h(x) = x^2+\rho^2$. Applying the power rule to $g(x)$ yields $g'(x) = \frac{5}{2}x^{\frac{3}{2}}$. Doing the same for $h(x)$, $h'(x) = 2x$. In all,

$$f'(x) = \frac{5}{2}(x^2+\rho^2)^{\frac{3}{2}}\cdot 2x = 5x(x^2+\rho^2)^{\frac{3}{2}}. \tag{3.23}$$

Moving on to the fraction as a whole, the quotient rule states, given a function $q(x) = \frac{r(x)}{s(x)}$,

$$q'(x) = \frac{r'(x)s(x) - r(x)s'(x)}{\big(s(x)\big)^2}.$$

In this situation, $q(x) = \frac{2x^2-\rho^2}{[x^2+\rho^2]^{\frac{5}{2}}}$, $r(x) = 2x^2 - \rho^2$, and $s(x) = f(x) = [x^2 + \rho^2]^{\frac{5}{2}}$. Applying the power rule to $r(x)$ yields $r'(x) = 4x$. Using the definition of $f'(x)$ given by equation 3.23 in the equation for $q'(x)$,

$$q'(x) = \frac{4x \cdot [x^2 + \rho^2]^{\frac{5}{2}} - (2x^2 - \rho^2) \cdot 5x\,(x^2 + \rho^2)^{\frac{3}{2}}}{\left([x^2 + \rho^2]^{\frac{5}{2}}\right)^2}.$$

Factoring out $x(x^2 + \rho^2)^{\frac{3}{2}}$ from the numerator and simplifying,

$$q'(x) = \frac{x(x^2 + \rho^2)^{\frac{3}{2}}[4(x^2 + \rho^2) - (2x^2 - \rho^2) \cdot 5]}{(x^2 + \rho^2)^5}$$

$$= \frac{x[4(x^2 + \rho^2) - 5(2x^2 - \rho^2)]}{(x^2 + \rho^2)^{\frac{7}{2}}}$$

$$= \frac{x[4x^2 + 4\rho^2 - 10x^2 + 5\rho^2]}{(x^2 + \rho^2)^{\frac{7}{2}}}$$

$$= \frac{x[9\rho^2 - 6x^2]}{(x^2 + \rho^2)^{\frac{7}{2}}}$$

$$= \frac{3x[3\rho^2 - 2x^2]}{(x^2 + \rho^2)^{\frac{7}{2}}}. \qquad (3.24)$$

Using this result in equation 3.22,

$$\frac{dB_x(\gamma, x, \rho)}{dx} = \frac{\mu_0 \gamma}{4\pi} \cdot \frac{3x[3\rho^2 - 2x^2]}{(x^2 + \rho^2)^{\frac{7}{2}}}. \tag{3.25}$$

Substituting equation 3.21 into equation 3.15,

$$\varepsilon = -vk \int_0^L \int_0^R \left(\frac{dB_x(m, x, \rho)}{dx} + \frac{dB_x(-m, x - L, \rho)}{dx} \right) \rho \, d\rho \, dx \tag{3.26}$$

Using the sum rule, this equation can be split into two parts such that

$$\varepsilon = -vk \cdot \big(\varepsilon_C(m, x) + \varepsilon_C(-m, x - L)\big) \tag{3.27}$$

where

$$\varepsilon_C(\gamma, x) = \int_0^L \int_0^R \frac{dB_x(\gamma, x, \rho)}{dx} \rho \, d\rho \, dx \tag{3.28}$$

Substituting the definition of $\frac{dB_x(\gamma,x,\rho)}{dx}$ given by equation 3.25 into equation 3.28,

$$\varepsilon_C(\gamma, x) = \int_0^L \int_0^R \frac{\mu_0 \gamma}{4\pi} \cdot \frac{3x[3\rho^2 - 2x^2]}{(x^2 + \rho^2)^{\frac{7}{2}}} \rho \, d\rho \, dx$$

$$= \frac{\mu_0 \gamma}{4\pi} \int_0^L \int_0^R \frac{3x[3\rho^2 - 2x^2]}{(x^2 + \rho^2)^{\frac{7}{2}}} \rho \, d\rho \, dx. \tag{3.29}$$

Solving the inner integral of equation 3.29 given by

$$\int_0^R \frac{3x[3\rho^2 - 2x^2]}{(x^2 + \rho^2)^{\frac{7}{2}}} \rho \, d\rho,$$

we can bring $3x$ in front of the integral to yield

$$3x\int_0^R \frac{\rho[3\rho^2 - 2x^2]}{(x^2+\rho^2)^{\frac{7}{2}}}\,d\rho\,. \tag{3.30}$$

We can use u substitution with a value $u = x^2 + \rho^2$ to begin solving the integral. This gives

$$\frac{du}{d\rho} = \frac{d}{d\rho}(x^2+\rho^2) = 2\rho,$$

leading to

$$d\rho = \frac{du}{2\rho}. \tag{3.31}$$

To substitute u into the numerator without any residual ρ, it can be seen that

$$3(x^2+\rho^2) - 5x^2 = 3\rho^2 - 2x^2.$$

This gives a resultant expression

$$3u - 5x^2 \tag{3.32}$$

which can be substituted into the numerator. Substituting equations 3.31 and 3.32 into equation 3.30 yields

$$\frac{3}{2}x\int_{x^2+0^2}^{x^2+R^2} \frac{3u-5x^2}{u^{\frac{7}{2}}}\,du\,. \tag{3.33}$$

Applying the sum rule and simplifying, this becomes

$$\frac{3}{2}x\left(\int \frac{3u}{u^{\frac{7}{2}}}du - \int \frac{5x^2}{u^{\frac{7}{2}}}du\right)\Bigg|_{x^2}^{x^2+R^2}$$

$$= \frac{3}{2}x\left(3\int \frac{1}{u^{\frac{5}{2}}}du - 5x^2\int \frac{1}{u^{\frac{7}{2}}}du\right)\Bigg|_{x^2}^{x^2+R^2}. \qquad (3.34)$$

Applying the power rule to the first term gives

$$3\int \frac{1}{u^{\frac{5}{2}}}du = 3\left(-\frac{2}{3u^{\frac{3}{2}}}\right) = -\frac{2}{u^{\frac{3}{2}}}. \qquad (3.35)$$

Applying the power rule to the second term gives

$$-5x^2\int \frac{1}{u^{\frac{7}{2}}}du = -5x^2\left(-\frac{2}{5u^{\frac{5}{2}}}\right) = \frac{2x^2}{u^{\frac{5}{2}}}. \qquad (3.36)$$

Substituting equations 3.35 and 3.36 into equation 3.34 and simplifying,

$$\frac{3}{2}x\left(-\frac{2}{u^{\frac{3}{2}}} + \frac{2x^2}{u^{\frac{5}{2}}}\right)\Bigg|_{x^2}^{x^2+R^2}$$

$$= \frac{3}{2}x\left(\frac{-2u + 2x^2}{u^{\frac{5}{2}}}\right)\Bigg|_{x^2}^{x^2+R^2}$$

$$= \frac{3}{2}x\left(\frac{-2(x^2+R^2) + 2x^2}{(x^2+R^2)^{\frac{5}{2}}} - \frac{-2x^2 + 2x^2}{(x^2)^{\frac{5}{2}}}\right)$$

$$= -\frac{3R^2x}{(x^2+R^2)^{\frac{5}{2}}}. \tag{3.37}$$

Substituting this value back into equation 3.29,

$$\varepsilon_C(\gamma, x) = \frac{\mu_0\gamma}{4\pi}\int_0^L \left(-\frac{3R^2x}{(x^2+R^2)^{\frac{5}{2}}}\right)dx.$$

Bringing $-3R^2$ in front of the integral, this becomes

$$\varepsilon_C(\gamma, x) = -3R^2\frac{\mu_0\gamma}{4\pi}\int_0^L \frac{x}{(x^2+R^2)^{\frac{5}{2}}}dx. \tag{3.38}$$

We can use u substitution with a value $u = x^2 + R^2$ to begin simplifying equation 3.38, giving

$$\frac{du}{dx} = \frac{d}{dx}(x^2+R^2) = 2x, \tag{3.39}$$

leading to

$$dx = \frac{du}{2x}. \tag{3.40}$$

Substituting equations 3.39 and 3.40 into equation 3.38 and simplifying yields

$$\begin{aligned}\varepsilon_C(\gamma, x) &= -3R^2\frac{\mu_0\gamma}{4\pi}\int_{R^2}^{L^2+R^2} \frac{x}{u^{\frac{5}{2}}}\frac{du}{2x} \\ &= -\frac{3R^2}{2}\cdot\frac{\mu_0\gamma}{4\pi}\int_{R^2}^{L^2+R^2}\frac{1}{u^{\frac{5}{2}}}du\,.\end{aligned}$$

Applying the power rule, this becomes

$$\varepsilon_C(\gamma, x) = -\frac{3R^2}{2} \cdot \frac{\mu_0 \gamma}{4\pi} \left(-\frac{2}{3u^{\frac{3}{2}}} \right) \Bigg|_{R^2}^{L^2+R^2} . \tag{3.41}$$

Substituting u back into equation 3.41 yields

$$\varepsilon_C(\gamma, x) = -\frac{3R^2}{2} \cdot \frac{\mu_0 \gamma}{4\pi} \left(-\frac{2}{3(x^2+R^2)^{\frac{3}{2}}} \right) \Bigg|_{0}^{L} . \tag{3.42}$$

This is the general form of $\varepsilon_c(\gamma, x)$ which can be used to find both terms of equation 3.27.

$\varepsilon_C(m, x)$ becomes

$$\varepsilon_c(m, x) = -\frac{3R^2}{2} \cdot \frac{\mu_0 m}{4\pi} \left(-\frac{2}{3(x^2+R^2)^{\frac{3}{2}}} \right) \Bigg|_{0}^{L}$$

$$= -\frac{3R^2}{2} \cdot \frac{\mu_0 m}{4\pi} \left(-\frac{2}{3(L^2+R^2)^{\frac{3}{2}}} + \frac{2}{3(R^2)^{\frac{3}{2}}} \right)$$

$$= \frac{\mu_0 m}{4\pi} \left(\frac{R^2}{(L^2+R^2)^{\frac{3}{2}}} - \frac{1}{R} \right) \tag{3.43}$$

while $\varepsilon_C(-m, x-L)$ becomes

$$\varepsilon_c(-m, x-L) = \frac{3R^2}{2} \cdot \frac{\mu_0 m}{4\pi} \left(-\frac{2}{3((x-L)^2+R^2)^{\frac{3}{2}}} \right) \Bigg|_{0}^{L}$$

$$= \frac{3R^2}{2} \cdot \frac{\mu_0 m}{4\pi} \left(-\frac{2}{3(R^2)^{\frac{3}{2}}} + \frac{2}{3(L^2+R^2)^{\frac{3}{2}}} \right)$$

$$= \frac{\mu_0 m}{4\pi}\left(\frac{R^2}{(L^2 + R^2)^{\frac{3}{2}}} - \frac{1}{R}\right). \tag{3.44}$$

Substituting equations 3.43 and 3.44 into equation 3.27,

$$\varepsilon = -vk \cdot \frac{\mu_0 m}{2\pi}\left(\frac{R^2}{(L^2 + R^2)^{\frac{3}{2}}} - \frac{1}{R}\right).$$

Substituting in the definition of k given by equation 3.3, this becomes

$$\varepsilon = -v \cdot \frac{\mu_0 m N}{L}\left(\frac{R^2}{(L^2 + R^2)^{\frac{3}{2}}} - \frac{1}{R}\right). \tag{3.45}$$

This equation will be used later to find the eddy current resulting from the movement of the train.

Calculation of Net Magnetic Field

The force experienced by the train which drives it forward in the coil is caused by the magnets' exposure to the magnetic field generated by the current flowing through the coil. We can use the Biot-Savart law to find the magnetic field at any point $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ generated by the current flowing through the coil between the two magnets. The Biot-Savart law states

$$\mathbf{B}(\mathbf{p}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{p}' \times (\mathbf{p} - \mathbf{p}')}{|\mathbf{p} - \mathbf{p}'|^3} dx' \tag{4.1}$$

(Fitzpatrick, The Biot-Savart law, 2006), where $\mathbf{p}'$ is a point representing the position of the infinitesimal section of coil used in the integration over the helix and

$$I = I_B + I_E \tag{4.2}$$

defines the net current through the coil, where I_B represents the current provided by the battery and I_E is the eddy current generated by the movement of the train through the coil. This operation can be conceptualized as summing the contributions of every infinitesimal section of wire along the coil to the magnetic field at the point $\mathbf{p}$. The coil can be modeled as a helix where the infinitesimal section $\mathbf{p}'$ of the helix can be found with

$$\mathbf{p}' = \begin{bmatrix} x' \\ -R\cos(kx') \\ -R\sin(kx') \end{bmatrix}, \tag{4.3}$$

where R is the radius of the coil and $k = \frac{2\pi}{s} = \frac{2\pi N}{L}$, as before. Within this calculation, $0 \leq x' \leq L$ since the integration is being calculated over the part of the helix through which current is flowing. In this expression, $d\mathbf{p}'$ is the derivative of $\mathbf{p}'$ with respect to x', which can be calculated as

$$\frac{d}{dx'}\mathbf{p}' = \frac{\mathrm{d}}{dx'}\begin{bmatrix} x' \\ -R\cos(kx') \\ -R\sin(kx') \end{bmatrix} = \begin{bmatrix} 1 \\ kR\sin(kx') \\ -kR\cos(kx') \end{bmatrix}. \tag{4.4}$$

We will be using this equation in the context of finding the magnetic field by the helical current present at the center of each dipole. Since the dipoles are located at $\mathbf{O}_{d_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{O}_{d_2} = \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix}$, we know that p_y and p_z will always be zero; thus, we can define $\mathbf{p}$ as

$$\mathbf{p} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}. \tag{4.5}$$

Equation 4.1 can be bounded and $\mathbf{p}$, $\mathbf{p}'$, and $d\mathbf{p}'$ replaced with their component forms from equations $4.4 - 4.6$ to give

$$\mathbf{B}(x) = \frac{\mu_0 I}{4\pi}\int_0^L \frac{\begin{bmatrix} 1 \\ kR\sin(kx') \\ -kR\cos(kx') \end{bmatrix} \times \left(\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x' \\ -R\cos(kx') \\ -R\sin(kx') \end{bmatrix} \right)}{\left\| \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x' \\ -R\cos(kx') \\ -R\sin(kx') \end{bmatrix} \right\|^3} dx'. \tag{4.6}$$

Beginning with the simplification of the numerator,

$$\begin{bmatrix} 1 \\ kR\sin(kx') \\ -kR\cos(kx') \end{bmatrix} \times \left(\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x' \\ -R\cos(kx') \\ -R\sin(kx') \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ kR\sin(kx') \\ -kR\cos(kx') \end{bmatrix} \times \begin{bmatrix} x - x' \\ R\cos(kx') \\ R\sin(kx') \end{bmatrix}$$

$$= \begin{bmatrix} kR^2\sin^2(kx') + kR^2\cos^2(kx') \\ -kR\cos(kx')(x - x') - R\sin(kx') \\ R\cos(kx') - kR\sin(kx')(x - x') \end{bmatrix}.$$

We will find later that the only component of **B** relevant to the problem is B_x. Continuing with the x-component only,

$$kR^2(\sin^2(kx') + \cos^2(kx')).$$

Using the Pythagorean identity, this can be further reduced to

$$kR^2. \tag{4.7}$$

Moving on to the simplification of the denominator,

$$\left\| \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x' \\ -R\cos(kx') \\ -R\sin(kx') \end{bmatrix} \right\|^3 = \left\| \begin{bmatrix} x - x' \\ R\cos(kx') \\ R\sin(kx') \end{bmatrix} \right\|^3.$$

We can use the distance formula to calculate the magnitude of the vector:

$$\sqrt{(x - x')^2 + R^2 \cos^2(kx') + R^2 \sin^2(kx')}^3$$

$$= [(x - x')^2 + R^2 \cos^2(kx') + R^2 \sin^2(kx')]^{\frac{3}{2}}$$

$$= [(x - x')^2 + R^2(\cos^2(kx') + \sin^2(kx'))]^{\frac{3}{2}}.$$

Again leveraging the Pythagorean identity, this becomes

$$[(x - x')^2 + R^2]^{\frac{3}{2}}. \tag{4.8}$$

Substituting equations 4.7 and 4.8 into equation 4.3,

$$B_x(x) = \frac{\mu_0 I}{4\pi} \int_0^L \frac{kR^2}{[(x - x')^2 + R^2]^{\frac{3}{2}}} dx'.$$

kR^2 can be brought in front of the integral to yield

$$B_x(x) = kR^2 \frac{\mu_0 I}{4\pi} \int_0^L \frac{1}{[(x - x')^2 + R^2]^{\frac{3}{2}}} dx'. \tag{4.9}$$

We can begin solving this integral with u-substitution. Given a value $u = x' - x$, then

$$\frac{du}{dx'} = \frac{d}{dx'}[x' - x] = 1.$$

Substituting these values into equation 4.9,

$$B_x(x) = kR^2 \frac{\mu_0 I}{4\pi} \int_{-x}^{L-x} \frac{1}{[u^2 + R^2]^{\frac{3}{2}}} du.$$

We can further substitute $v = \arctan\left(\frac{u}{R}\right)$ to obtain $du = R\sec^2(v)\,dv$, yielding

$$B_x(x) = kR^2 \frac{\mu_0 I}{4\pi} \int_{\arctan\left(\frac{-x}{R}\right)}^{\arctan\left(\frac{L-x}{R}\right)} \frac{R\sec^2(v)}{[R^2\tan^2(v) + R^2]^{\frac{3}{2}}} dv.$$

$$= kR^2 \frac{\mu_0 I}{4\pi} \int_{\arctan\left(\frac{-x}{R}\right)}^{\arctan\left(\frac{L-x}{R}\right)} \frac{R\sec^2(v)}{[R^2(\tan^2(v) + 1)]^{\frac{3}{2}}} dv.$$

Recognizing another Pythagorean identity, this becomes

$$B_x(x) = kR^2 \frac{\mu_0 I}{4\pi} \int_{\arctan\left(\frac{-x}{R}\right)}^{\arctan\left(\frac{L-x}{R}\right)} \frac{R\sec^2(v)}{[R^2\sec^2(v)]^{\frac{3}{2}}} dv$$

$$= kR^2 \frac{\mu_0 I}{4\pi} \int_{\arctan\left(\frac{-x}{R}\right)}^{\arctan\left(\frac{L-x}{R}\right)} \frac{1}{R^2\sec(v)} dv$$

$$= k \frac{\mu_0 I}{4\pi} \int_{\arctan\left(\frac{-x}{R}\right)}^{\arctan\left(\frac{L-x}{R}\right)} \cos(v)\,dv$$

$$= k \frac{\mu_0 I}{4\pi} (\sin(v))\Big|_{\arctan\left(\frac{-x}{R}\right)}^{\arctan\left(\frac{L-x}{R}\right)}.$$

Substituting the above definition of v,

$$B_x(x) = k\frac{\mu_0 I}{4\pi}\left(\sin\left(\arctan\left(\frac{u}{R}\right)\right)\right)\Big|_{-x}^{L-x}.$$

Substituting the above definition of u,

$$B_x(x) = k\frac{\mu_0 I}{4\pi}\left(\sin\left(\arctan\left(\frac{x' - x}{R}\right)\right)\right)\Bigg|_{0}^{L}$$

$$= k\frac{\mu_0 I}{4\pi}\cdot\left(\sin\left(\arctan\left(\frac{L-x}{R}\right)\right) - \sin\left(\arctan\left(\frac{-x}{R}\right)\right)\right).$$

Using the identity

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}},$$

$$B_x(x) = k\frac{\mu_0 I}{4\pi}\cdot\left(\frac{\frac{L-x}{R}}{\sqrt{1+\left(\frac{L-x}{R}\right)^2}} - \frac{\frac{-x}{R}}{\sqrt{1+\left(\frac{-x}{R}\right)^2}}\right)$$

$$= k\frac{\mu_0 I}{4\pi}\cdot\left(\frac{L-x}{R\sqrt{1+\frac{(L-x)^2}{R^2}}} + \frac{x}{R\sqrt{1+\frac{x^2}{R^2}}}\right)$$

$$= k\frac{\mu_0 I}{4\pi}\cdot\left(\frac{L-x}{R\sqrt{\frac{R^2+(L-x)^2}{R^2}}} + \frac{x}{R\sqrt{\frac{R^2+x^2}{R^2}}}\right)$$

$$= k\frac{\mu_0 I}{4\pi}\cdot\left(\frac{L-x}{\sqrt{R^2+(L-x)^2}} + \frac{x}{\sqrt{R^2+x^2}}\right). \qquad (4.10)$$

Calculation of Theoretical Terminal Velocity

The terminal velocity of the train is reached when the net force on the train is zero. This is when the force on the train generated by the magnetic field resulting from the current through the coil is equal to the combined force of friction and the weight of the hanging weight opposing the motion of the train, as expressed by

$$F_{x_{total}} - M_T g \mu_k - M_W g = 0, \tag{5.1}$$

where $F_{x_{total}}$ is the driving force on the train from the helical current, M_T is the mass of the train, g is the acceleration of free fall, μ_k is the coefficient of kinetic friction of the system, and M_W is the mass of the hanging weight.

$F_{x_{total}}$ can be calculated using the gradient of the potential energy of the magnets in the magnetic field. The potential energy of a magnetic dipole with magnetic moment $\boldsymbol{\gamma}$ in a magnetic field **B** can be found with

$$U(\boldsymbol{\gamma}, \mathbf{p}) = -\boldsymbol{\gamma} \cdot \mathbf{B}(\mathbf{p}) \tag{5.2}$$

(Griffiths, 2018, p. 291). The force on this dipole can be expressed as

$$\mathbf{F}(\boldsymbol{\gamma}, \mathbf{p}) = -\nabla \mathbf{U}(\boldsymbol{\gamma}, \mathbf{p}) \tag{5.3}$$

(Tegmark, 2014) where $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$ is a vector of derivative operators. Substituting equation 5.2 into equation 5.3, this becomes

$$\mathbf{F}(\boldsymbol{\gamma}, \mathbf{p}) = \nabla\big(\boldsymbol{\gamma} \cdot \mathbf{B}(\mathbf{p})\big). \tag{5.4}$$

Since we only have a definition for B_x, we can only find F_x, and ∇ can be reduced to $\frac{\partial}{\partial x}$. The net force on the train will be a combination of the forces experienced by dipoles d_1 and d_2, which can be found using values of $m_{d_1} = m$ and $p_{d_{1}x} = 0$ and $m_{d_2} = -m$ and $p_{d_{2}x} = L$ respectively, as expressed by

$$F_{x_{total}} = F_x(m, 0) + F(-m, L). \tag{5.5}$$

Since the dipoles have magnetic moments with y- and z-values of 0, we can redefine equation 5.4 as

$$F_x(\gamma, x) = \frac{\partial}{\partial x}\big(\gamma \cdot B_x(x)\big).$$

Substituting in the definition of $B_x(x)$ provided by equation 4.10,

$$F_x(\gamma, x) = \frac{\partial}{\partial x}\left(\gamma \cdot k\frac{\mu_0 I}{4\pi} \cdot \left(\frac{L - x}{\sqrt{R^2 + (L - x)^2}} + \frac{x}{\sqrt{R^2 + x^2}}\right)\right)$$

$$= \gamma \cdot k\frac{\mu_0 I}{4\pi} \cdot \frac{\partial}{\partial x}\left(\frac{L - x}{\sqrt{R^2 + (L - x)^2}} + \frac{x}{\sqrt{R^2 + x^2}}\right). \tag{5.6}$$

Using the sum rule,

$$\frac{\partial}{\partial x}\left(\frac{L - x}{\sqrt{R^2 + (L - x)^2}} + \frac{x}{\sqrt{R^2 + x^2}}\right) = \frac{\partial}{\partial x}\left(\frac{L - x}{\sqrt{R^2 + (L - x)^2}}\right) + \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{R^2 + x^2}}\right).$$

Beginning with the first term, using the quotient rule,

$$\frac{\partial}{\partial x}\left(\frac{L-x}{\sqrt{R^2+(L-x)^2}}\right)=\frac{\frac{(L-x)^2}{\sqrt{R^2+(L-x)^2}}-\sqrt{R^2+(L-x)^2}}{R^2+(L-x)^2}$$

$$=-\frac{R^2}{(R^2+(L-x)^2)^{\frac{3}{2}}}. \tag{5.7}$$

Doing the same with the second term,

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{R^2+x^2}}\right)=\frac{\sqrt{R^2+x^2}-\frac{x^2}{\sqrt{R^2+x^2}}}{R^2+x^2}$$

$$=\frac{R^2}{(R^2+x^2)^{\frac{3}{2}}}. \tag{5.8}$$

Substituting equations 5.7 and 5.8 into equation 5.6 and expanding k with its definition from equation 3.3,

$$F_x(\gamma,x)=R^2\cdot\gamma\cdot\frac{\mu_0 IN}{L}\cdot\left(\frac{1}{(R^2+x^2)^{\frac{3}{2}}}-\frac{1}{(R^2+(L-x)^2)^{\frac{3}{2}}}\right). \tag{5.9}$$

Using equation 5.9 in equation 5.5,

$$F_{x_{total}}=R^2\cdot m\cdot\frac{\mu_0 IN}{L}\cdot\left(\left(\frac{1}{(R^2)^{\frac{3}{2}}}-\frac{1}{(R^2+L^2)^{\frac{3}{2}}}\right)-\left(\frac{1}{(R^2+L^2)^{\frac{3}{2}}}-\frac{1}{(R^2)^{\frac{3}{2}}}\right)\right)$$

$$=R^2\cdot m\cdot\frac{\mu_0 IN}{L}\cdot\left(\frac{1}{R^3}-\frac{1}{(R^2+L^2)^{\frac{3}{2}}}-\frac{1}{(R^2+(L)^2)^{\frac{3}{2}}}+\frac{1}{R^3}\right)$$

$$= 2m \cdot \frac{\mu_0 IN}{L} \cdot \left(\frac{1}{R} - \frac{R^2}{(R^2 + L^2)^{\frac{3}{2}}}\right) \tag{5.10}$$

Substituting equation 5.10 into equation 5.1 and rearranging for I,

$$2m \cdot \frac{\mu_0 IN}{L} \cdot \left(\frac{1}{R} - \frac{R^2}{(R^2 + L^2)^{\frac{3}{2}}}\right) - M_T g \mu_k - M_W g = 0$$

$$2m \cdot \frac{\mu_0 IN}{L} \cdot \left(\frac{1}{R} - \frac{R^2}{(R^2 + L^2)^{\frac{3}{2}}}\right) = M_T g \mu_k + M_W g$$

$$I \cdot \left(\frac{(R^2 + L^2)^{\frac{3}{2}} - R^3}{R(R^2 + L^2)^{\frac{3}{2}}}\right) = \frac{L}{\mu_0 mN}(M_T g \mu_k + M_W g)$$

$$I = \left(\frac{R(R^2 + L^2)^{\frac{3}{2}}}{(R^2 + L^2)^{\frac{3}{2}} - R^3}\right)\left(\frac{L}{\mu_0 mN}(M_T g \mu_k + M_W g)\right).$$

Substituting in the definition of I provided in equation 4.2,

$$I_B + I_E = \left(\frac{R(R^2 + L^2)^{\frac{3}{2}}}{(R^2 + L^2)^{\frac{3}{2}} - R^3}\right)\left(\frac{L}{\mu_0 mN}(M_T g \mu_k + M_W g)\right).$$

Substituting in equation 3.1 and rearranging for ε, this becomes

$$I_B + \frac{\varepsilon}{R_T} = \left(\frac{R(R^2 + L^2)^{\frac{3}{2}}}{(R^2 + L^2)^{\frac{3}{2}} - R^3}\right)\left(\frac{L}{\mu_0 mN}(M_T g \mu_k + M_W g)\right)$$

$$\varepsilon = R_T\left(\left(\frac{R(R^2+L^2)^{\frac{3}{2}}}{(R^2+L^2)^{\frac{3}{2}}-R^3}\right)\left(\frac{L}{\mu_0 mN}(M_T g\mu_k + M_W g)\right) - I_B\right).$$

Substituting in equation 3.45 and rearranging for v,

$$-v\cdot\frac{\mu_0 mN}{L}\left(\frac{R^2}{(L^2+R^2)^{\frac{3}{2}}}-\frac{1}{R}\right) = R_T\left(\left(\frac{R(R^2+L^2)^{\frac{3}{2}}}{(R^2+L^2)^{\frac{3}{2}}-R^3}\right)\left(\frac{L}{\mu_0 mN}(M_T g\mu_k + M_W g)\right) - I_B\right)$$

$$v = -R_T\left(\frac{L}{\mu_0 mN}\right)\left(\frac{R(R^2+L^2)^{\frac{3}{2}}}{(R^2+L^2)^{\frac{3}{2}}-R^3}\right)\left[\left(\frac{L}{\mu_0 mN}\right)\left(\frac{R(R^2+L^2)^{\frac{3}{2}}}{(R^2+L^2)^{\frac{3}{2}}-R^3}\right)(M_T g\mu_k + M_W g) - I_B\right]$$

$$v = -\left(\frac{L}{\mu_0 mN}\right)\left(\frac{R(R^2+L^2)^{\frac{3}{2}}}{(R^2+L^2)^{\frac{3}{2}}-R^3}\right)\left[R_T\left(\frac{L}{\mu_0 mN}\right)\left(\frac{R(R^2+L^2)^{\frac{3}{2}}}{(R^2+L^2)^{\frac{3}{2}}-R^3}\right)(M_T g\mu_k + M_W g) - V\right]$$

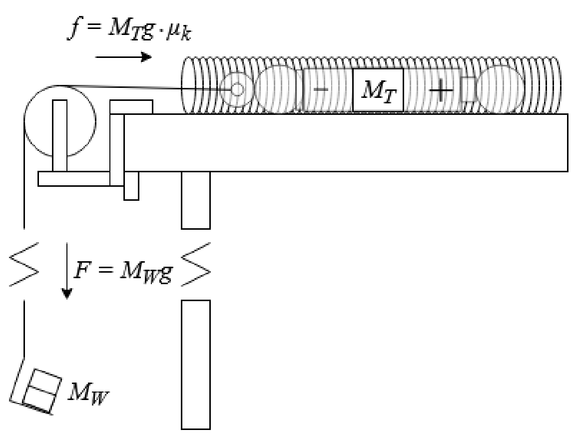
where V is the voltage across the battery.

Experiment Procedure and Collected Data

V, R_T, and μ_k must be determined experimentally. The voltage can be directly measured with a multimeter by touching the probes to either end of the battery while the train is held in contact with inside of the coil without moving. The total resistance can be found with knowledge of the current flowing through the coil when the train is held still. To create an opening in the circuit, a magnet can be brought out of contact with the battery by placing an insulating washer between the conductive washer and the magnet and attaching the multimeter probes to the conductive washer and adjacent magnet so as to close the circuit. The washers extend the distance L separating the magnets by a negligible amount. Ohm's law can then be used to find the resistance of the circuit.

The coefficient of kinetic friction can be calculated by using a depleted battery, performing the same procedure with the conductive and insulating washers to open the circuit and reduce the presence of eddy currents, and dragging the train through the coil with a hanging weight and measuring its acceleration. The acceleration of the train is measured by configuring the system such that the movement of the string attaching the washer to the hanging weight spins a pulley on the side of the table, the acceleration of which is measured by an attached photogate, as illustrated in *Diagram 3*.

Diagram 3: Configuration of μ_k experiment



As *Diagram 3* illustrates, the hanging weight exerts a force

$$F = M_W g \tag{6.1}$$

on the train which is opposed by a frictional force

$$f = M_T g \cdot \mu_k. \tag{6.2}$$

The acceleration of the system can be found with

$$a = \frac{F - f}{M_T + M_W} \tag{6.3}$$

Substituting equations 6.1 and 6.2 into equation 6.3 and rearranging for μ_k,

$$a = \frac{M_W g - M_T g \cdot \mu_k}{M_T + M_W}$$

$$a(M_T + M_W) = M_W g - M_T g \cdot \mu_k$$

$$a(M_T + M_W) - M_W g = -M_T g \cdot \mu_k$$

$$\mu_k = \frac{M_W g - a(M_T + M_W)}{M_T g}. \tag{6.4}$$

Using values of $M_T = 0.0820$ kg and $M_W = 0.05$ kg, I took ten measurements of the average acceleration of the system and calculated the corresponding coefficient of kinetic friction using equation 6.4. The results are shown in *Table 1*.

Table 1: Results of μ_k *experiment with* a *and corresponding* μ_k

a (m/s/s)	μ_k
1.64	0.338
1.64	0.338
1.55	0.352
1.65	0.352
1.55	0.339
1.63	0.352
1.55	0.352
1.55	0.362
1.49	0.362
1.49	0.348

The average of these values is $\mu_k = 0.348$, which is the value used in all relevant calculations.

To verify that this model is an accurate description of the phenomenon, I measured the maximum velocity of the train in its transit ten times. The method with which the velocity of the train was measured is identical to that described above, although the 0.05 kg weight was replaced with a 0.003 kg weight and the additional washers were removed to allow contact between both magnets and the battery. I will use values of $m = 5.7\ \mathrm{A\ m^2}$, $R = 0.012\ \mathrm{m}$; $k = \frac{2\pi}{s} = \frac{2\pi N}{L}$ where $N = 35.0$ turns and $L = 0.089\ \mathrm{m}$; $M = 0.0820\ \mathrm{kg}$; $g = 9.81\frac{\mathrm{m}}{\mathrm{s}^2}$; $\mu_k = 0.348$; and $M_W = 0.003\ \mathrm{kg}$ in the calculations.

Unfortunately, the percent error of the predicted v is extremely variable and often very high, as demonstrated in *Table 2*.

Table 2: Results of v *experiment*

V (V)	I (A)	Calculated R_T (Ω)	Predicted v (m/s)	Observed v (m/s)	Percent error
0.2725	3.564	0.0765	0.730	0.54	35.1
0.2777	2.812	0.0988	0.625	0.51	22.6
0.2276	2.392	0.0952	0.432	0.54	20.1
0.2511	2.673	0.0939	0.539	0.44	22.5
0.2394	2.653	0.0902	0.510	0.42	21.4
0.2234	2.696	0.0829	0.483	0.47	2.9
0.2630	2.274	0.116	0.466	0.52	10.3
0.2452	2.139	0.115	0.396	0.48	17.5
0.2568	2.056	0.125	0.387	0.5	22.5
0.1995	1.896	0.105	0.254	0.2	27.2

There was an abundance of noise in the data due to poor contact between the multimeter and the train and the unideal quality of construction of the coil.

Conclusion and Evaluation

The motion of the train has been modelled in a general form which is applicable to most configurations of the experiment. An equation was derived with which the terminal velocity of the train can be calculated with sufficient knowledge of the physical characteristics of the system. The experimental portion of this essay has provided weak support for the validity of the developed model, with a troubling inconsistency between the predicted and observed terminal velocities of the train.

There are many sources of error in this experiment which may have produced this inconsistency. Firstly, the collection of the voltage difference between the terminals of the battery and the current flowing through the system was very unstable, and the readings on the multimeter varied wildly, leading to considerable uncertainty in the data collected. Compounding these inaccuracies, the dimensions of the coil and the number of turns of the coil between the two magnets is variable throughout the coil. Although I constructed the coil to the best of my ability, variability in the form of the coil is unavoidable when coiled by hand, which is likely the main source of the noise in the observed velocity of the train. In addition, the train bounces along the coil during transit rather than maintaining constant contact with it, the effect of which was not considered in the model. Due to the number of variables which influence the terminal velocity of the train, there were many opportunities for errors in measurement to accumulate and alter the produced value.

There are also many modifications to the model which can be made to improve its validity. It was assumed that the train is perfectly centered in the track at all times; however, the train sits below the axis of the coil and is not perfectly still in any direction, which was not considered in the development of this model. The ferromagnetic properties of the battery likely have an effect on the strength of the magnetic field generated by both the helical current and either magnet much like an iron-core solenoid, but the procedure for measuring the relative magnetic permeability of a mixed substance like that found in a battery was too complex to be explored in this paper. Similarly, the washer attached to the trailing magnet is ferromagnetic and thus augments the magnetic field of the magnet it is attached to; unfortunately, the calculations for determining the magnetic field resulting from this interaction are outside the scope of this paper. In addition, neither the y-component nor the z-component of the motion of the train were considered in the calculation of the train's terminal velocity; the vertical force may have increased or decreased the amount of friction experienced by the train, and the lateral movement of the train through the coil may have generated additional friction which the model developed in this paper fails to take into account.

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